

Collective excitations of a degenerate Fermi vapour in a magnetic trap

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Abstract. We evaluate the small-amplitude excitations of a spin-polarized vapour of Fermi atoms confined inside a harmonic trap. The dispersion law $\omega = \omega_f[\ell + 4n(n + \ell + 2)/3]^{1/2}$ is obtained for the vapour in the collisional regime inside a spherical trap of frequency ω_f , with n the number of radial nodes and ℓ the orbital angular momentum. The low-energy excitations are also treated in the case of an axially symmetric harmonic confinement. The collisionless regime is discussed with main reference to a Landau-Boltzmann equation for the Wigner distribution function: this equation is solved within a variational approach allowing an account of non-linearities. A comparative discussion of the eigenmodes of oscillation for confined Fermi and Bose vapours is presented in an Appendix.

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1 Introduction

The achievement of condensation in confined vapours of bosonic alkali atoms [1–3] and more recently in vapours of hydrogen [4] has been giving great stimulus to the study of many-body and quantum statistical effects in dilute quantal fluids. In particular, the observation of small-amplitude shape-deformation modes in these bosonic condensates at very low temperature [5,6] has provided a crucial test for the theory as developed by many authors (see for instance [7–9]). Trapping of fermionic species has also been reported for ⁶Li [10] and ⁴⁰K [11]. Cooling of these Fermi gases into the regime of quantal degeneracy still remains to be achieved experimentally, but one may already anticipate that the observation of the dynamic response of the vapour of fermionic atoms to modulations of the trap will provide an important method of diagnostics. It is to the theoretical study of the elementary excitations in such dilute normal Fermi fluids at very low temperature that the present work is addressed.

We assume that the vapour has reached thermal equilibrium inside a magnetic trap and regard it as a dilute, spin-polarized Fermi fluid. The s -wave collisions between pairs of atom are inhibited by the Pauli principle, so that to leading order only p -wave scattering and dipole-dipole magnetic interactions are important. However, these interaction effects are very small at the temperatures of present interest [12,13]. The confined vapour may then be treated

as a non-interacting Fermi gas, with resistance to shape deformation being provided by exchange in the regime of quantal degeneracy.

In Section 2 we start treating an inhomogeneous gas composed of (a large number of) non-interacting fermions from the general viewpoint of the equation of motion for the Wigner distribution function and of its projections yielding the conservation laws in the form of quantal hydrodynamic equations [14]. On neglecting temperature fluctuations, the kinetic stress tensor contains the essential physical input for the study of both the equilibrium density profile and the dynamics of density fluctuations. In turn, the kinetic stress tensor is simply related to the local Pauli pressure and to the velocity field under two assumptions: (i) the inhomogeneity is weak on the length scale set by the inverse Fermi wavenumber, and (ii) local equilibrium is maintained in the dynamics of the vapour by frequent collisions against external scatterers (see the discussion given at the beginning of Sect. 2.2 and of Sect. 5 below). For the equilibrium state we recover from this approach the Thomas-Fermi form of the density profile already found by Mølmer [15] for fermions in a boson-fermion mixture at zero temperature. We proceed from here in Section 3 to evaluate the eigenmodes of small-amplitude oscillations for the Fermi vapour at zero temperature in a spherically symmetric trap. In an Appendix we show how one may recover within the same approach the results of Stringari [7] for the collective excitations of a Bose condensate in the strong coupling regime and comparatively discuss the nature of the eigenmode solutions for the Fermi gas and for the Bose condensate.

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In Section 4 we extend the discussion of the small-amplitude vibrations of an inhomogeneous Fermi gas to the case of confinement within an axially symmetric harmonic trap and give explicit results for the low-lying eigenmodes. In Section 5 we consider instead the dynamics of a Fermi gas under spherical confinement in the case where collisions have a negligible effect on the deviations from equilibrium. We make brief reference to the zero-sound excitation of the homogeneous Fermi gas in such a collisionless regime and then focus on a variational treatment of a Landau-Boltzmann equation for the Wigner distribution function. Finally, Section 6 gives a brief summary of our main results and offers some concluding remarks.

2 Small-amplitude excitations in the collisional regime

We describe the inhomogeneous spin-polarized fermion cloud by the one-particle density matrix

$$\rho(\mathbf{x}, \mathbf{x}'; t) = \langle \hat{\Psi}^\dagger(\mathbf{x}, t) \hat{\Psi}(\mathbf{x}', t) \rangle \quad (2.1)$$

where $\hat{\Psi}$ and $\hat{\Psi}^\dagger$ are the fermionic field operators satisfying the usual anticommutation relations and the brackets denote the expectation value in the equilibrium state. We have suppressed the spin index, which is frozen because the magnetically trapped fermions are taken to belong to a single hyperfine level. The equation of motion for the density matrix of non-interacting fermions in an external potential $V_{\text{ext}}(\mathbf{x}, t)$ is (see for instance Singwi and Tosi [16])

$$[i\hbar\partial_t - (\hbar^2/2m)(\nabla_{\mathbf{x}}^2 - \nabla_{\mathbf{x}'}^2) + V_{\text{ext}}(\mathbf{x}, t) - V_{\text{ext}}(\mathbf{x}', t)]\rho(\mathbf{x}, \mathbf{x}'; t) = 0. \quad (2.2)$$

We shall take $V_{\text{ext}}(\mathbf{x}, t)$ as the sum of a harmonic static potential $V_0(\mathbf{x})$ and of a perturbing dynamic potential $V_1(\mathbf{x}, t)$.

The generalized hydrodynamic equations expressing the conservation laws are obtained by expanding equation (2.2) in powers of $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ around the diagonal $\mathbf{R} = (\mathbf{x} + \mathbf{x}')/2$. We introduce the Wigner distribution function $f_{\text{p}}(\mathbf{R}, t)$, which is related to $\rho(\mathbf{x}, \mathbf{x}'; t)$ by

$$f_{\text{p}}(\mathbf{R}, t) = \int d\mathbf{r} \rho(\mathbf{R} + \mathbf{r}/2, \mathbf{R} - \mathbf{r}/2; t) \exp(i\mathbf{p} \cdot \mathbf{r}/\hbar), \quad (2.3)$$

and its successive moments giving the density of particles $n(\mathbf{R}, t)$,

$$n(\mathbf{R}, t) = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} f_{\text{p}}(\mathbf{R}, t), \quad (2.4)$$

the current density $\mathbf{j}(\mathbf{R}, t)$,

$$\mathbf{j}(\mathbf{R}, t) = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \frac{\mathbf{p}}{m} f_{\text{p}}(\mathbf{R}, t) \quad (2.5)$$

and the kinetic stress tensor $\Pi_{ij}(\mathbf{R}, t)$,

$$\Pi_{ij}(\mathbf{R}, t) = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \frac{p_i p_j}{m} f_{\text{p}}(\mathbf{R}, t). \quad (2.6)$$

Then by the indicated expansion procedure we obtain the conservation laws in the form

$$\partial_t n(\mathbf{R}, t) = -\nabla \cdot \mathbf{j}(\mathbf{R}, t) \quad (2.7)$$

and

$$m\partial_t \mathbf{j}(\mathbf{R}, t) = -\nabla \cdot \Pi(\mathbf{R}, t) - n(\mathbf{R}, t) \nabla V_{\text{ext}}(\mathbf{R}, t). \quad (2.8)$$

We consider below only isothermal fluctuations and therefore omit to write down the energy conservation equation.

2.1 Approximations on the kinetic stress tensor

The kinetic stress tensor of an homogeneous fluid is related to the pressure P and to the velocity field \mathbf{v} by

$$\Pi_{ij} = P\delta_{ij} + nmv_i v_j. \quad (2.9)$$

For the ideal spin-polarized Fermi gas at zero temperature we have

$$P = 2E/3V = 3n\varepsilon_f/5 \quad (2.10)$$

where E is the ground-state energy and

$$\varepsilon_f = \hbar^2(6\pi^2 n)^{2/3}/2m$$

is the Fermi energy. On the other hand, the second term on the RHS of equation (2.9) can be dropped in the linear regime.

These results can be used to obtain the kinetic stress tensor of the inhomogeneous Fermi fluid if the inhomogeneity is sufficiently weak to allow a local-density approximation, *i.e.*

$$\Pi_{ij}(\mathbf{R}, t) = P(\mathbf{R}, t)\delta_{ij} = \frac{2}{5}A(n(\mathbf{R}, t))^{5/3}\delta_{ij} \quad (2.11)$$

where $A = \hbar^2(6\pi^2)^{2/3}/2m$. More explicitly, we are assuming that the length scale for the variation of the density profile in space is large relative to the inverse Fermi wavenumber k_f^{-1} .

We remark that insertion of equation (2.11) into equations (2.7, 2.8) and reduction to the case of a static external potential yields the equation for the equilibrium density profile $n_0(\mathbf{R})$ in the form

$$\frac{2}{5}A\nabla[n_0(\mathbf{R})]^{5/3} + n_0(\mathbf{R})\nabla V_0(\mathbf{R}) = 0. \quad (2.12)$$

The solution of equation (2.12) is the Thomas-Fermi profile for the confined Fermi vapour,

$$n_0(\mathbf{R}) = \vartheta(\varepsilon_f - V_0(\mathbf{R}))[(\varepsilon_f - V_0(\mathbf{R}))/A]^{3/2} \quad (2.13)$$

(see for instance Mølmer [15]). The profile vanishes continuously at a radius R_f , say.

2.2 Equation of motion for small amplitude fluctuations

With regard to the dynamics of the fermionic cloud, the approximation (2.11) is most suitable for treating a “collisional” regime in which the relaxation of the fluctuations is rapid on the time scale set by the frequency of the driving potential. We shall have to return on this point in Section 5 below. Here we proceed to study the dynamics of the fermion cloud for small distortions of its density profile around the equilibrium Thomas-Fermi profile given in equation (2.13).

If we denote by $n_1(\mathbf{R}, t)$ such a small-amplitude density fluctuation, equations (2.7, 2.8, 2.11) can be linearized and combined to yield the equation of motion

$$\partial_t^2 n_1(\mathbf{R}, t) = \frac{1}{m} \nabla_{\mathbf{R}} \cdot \left\{ n_1(\mathbf{R}, t) \nabla_{\mathbf{R}} V_0(\mathbf{R}) + \frac{2}{3} A \nabla_{\mathbf{R}} [n_0^{2/3}(\mathbf{R}) n_1(\mathbf{R}, t)] \right\}. \quad (2.14)$$

Equation (2.14) holds at resonance, since we have dropped the external perturbation drive. In the homogeneous limit ($V_0(\mathbf{R}) \rightarrow 0$) it describes sound propagation at a speed c given by $c = (2\varepsilon_f/3m)^{1/2}$ – a well-known result for first sound in the ideal Fermi gas [17]. In this so-called collisional regime the local compressions and expansions of the homogeneous fluid which are induced by the sound wave are reflected in spherically symmetric oscillations of the diameter of the Fermi sphere.

3 Small oscillations in a spherical trap

We determine in this section the solution of equation (2.14) for the case of a spherically symmetric harmonic confinement, *i.e.* $V_0(\mathbf{R}) = m\omega_f^2 R^2/2$.

We Fourier-transform equation (2.14) with respect to time and rescale all lengths in units of the harmonic oscillator length $a_{\text{ho}} = (\hbar/m\omega_f)^{1/3}$ by setting $\mathbf{R} = a_{\text{ho}}\mathbf{x}$. We get

$$3\omega_f^2 n_1(\mathbf{x}, \omega) = -\omega_f^2 \nabla_{\mathbf{x}} \cdot [n_1(\mathbf{x}, \omega) \nabla_{\mathbf{x}} (x^2/2)] + (X^2 - x^2) \nabla_{\mathbf{x}} n_1(\mathbf{x}, \omega), \quad (3.1)$$

where $X = (2\varepsilon_f/m\omega_f^2 a_{\text{ho}}^2)^{1/2}$ is the scaled radius of the fermionic cloud. We search for spherical solutions of equation (3.1) having the form

$$n_1(\mathbf{x}, \omega) = \vartheta(X - x) x^\ell F(x/X) Y_\ell^m(\theta, \varphi), \quad (3.2)$$

with $Y_\ell^m(\theta, \varphi)$ being the spherical harmonics. Clearly, we have imposed on the eigenmode solutions the condition that they be non-vanishing only inside the equilibrium cloud radius X . In addition we shall impose that the solutions vanish continuously at the cloud boundary $x = X$, as a consequence of Fermi statistics giving a high cost in kinetic energy to rapid variations of the density profile in space. This boundary condition on the eigenmodes of

vibration for a trapped Fermi gas is further discussed in the Appendix and is contrasted there with that imposed *e.g.* by Stringari [7] on those of a trapped (interacting) Bose gas, where considerations of kinetic energy only play a secondary role.

After setting $y = x/X$ and substituting the expression (3.2) into equation (3.1) we find the differential equation satisfied by $F(y)$,

$$(1 - y^2) \frac{d^2 F(y)}{dy^2} + \frac{2(\ell + 1) - (2\ell + 3)y^2}{y} \frac{dF(y)}{dy} + [3 + 3(\omega/\omega_f)^2 - \ell] F(y) = 0 \quad (3.3)$$

with the boundary condition $F(1) = 0$. We adopt the Fuchs method for solving an ordinary differential equation in a series form around regular singular points [18]. Using the above-mentioned condition of continuity at the boundary and the fact that $F(y)$ is an even function of its argument, the solutions of equation (3.3) are

$$F(y) = (1 - y^2)^{1/2} \sum_{k=0}^{\infty} \alpha_k (1 - y^2)^k \quad (3.4)$$

and the coefficients α_k satisfy the recurrence relation

$$\alpha_{k+1} = \frac{(2k + 1)(2k + 2\ell + 3) - [3 + 3(\omega/\omega_f)^2 - \ell]}{2(k + 1)(2k + 3)} \alpha_k. \quad (3.5)$$

Finally, the dispersion relation for the normal modes is obtained by imposing that the solutions (3.4, 3.5) reduce to polynomials, *i.e.* $\alpha_{n+1} = 0$ for an integer n representing the number of internal nodes of the density fluctuation profile. This yields

$$\omega(n, \ell) = \omega_f \left[\ell + \frac{4}{3} n(n + \ell + 2) \right]^{1/2}. \quad (3.6)$$

The dispersion relation (3.6) correctly displays the sloshing-mode solution $\omega = \omega_f$ at $(n = 0, \ell = 1, m = 0)$, in agreement with the generalized Kohn theorem [19].

It is also interesting to notice that the eigenfrequencies of the surface modes at $n = 0$ are the same as those found by Stringari [7] for a Bose condensate in a spherical trap and that the $\ell = 0$ modes resemble those found for a Fermi superfluid by Baranov and Petrov [20].

4 Extension to axially symmetric magnetic confinement

We evaluate in this section the low-lying solutions of equation (2.14) in the case of an axially symmetric harmonic confinement, which is more directly relevant in regard to experiment. The confining potential is chosen as

$$V_0(\rho, z) = \frac{1}{2} m\omega_f^2 (\rho^2 + \lambda^2 z^2) \quad (4.1)$$

where $\rho = (x^2 + y^2)^{1/2}$ is the radial coordinate and λ is the anisotropy parameter.

In the axially symmetric case only the L_z component of the angular momentum along the z -direction is conserved, whereas ℓ is no longer a good quantum number. However, for the low-lying modes we can still start from a spherical base as in equation (3.2) and suitably modify it so as to account for the cylindrical symmetry of the problem. As we shall see through an example below, the consequence of the reduced symmetry is a coupling between some of the modes which are characterized by different values of ℓ in the spherical base.

We give here the explicit results for the eigenfrequencies and the density-fluctuation profiles in some instances of dipolar and quadrupolar modes. In the expressions reported below, all lengths are still scaled in units of $a_{\text{ho}} = (\hbar/m\omega_f)^{1/2}$.

4.1 Dipole sloshing mode along the z -direction

Starting from the spherical-base solution corresponding to $(n = 0, \ell = 1, m = 0)$, we construct the density profile

$$n_1^{(0,1,0)}(\rho, z; \omega) = (1 - \rho^2 - \lambda^2 z^2)^{1/2} \rho. \quad (4.2)$$

The corresponding frequency is $\omega = \lambda\omega_f$, in accord with the generalized Kohn theorem [19].

4.2 Quadrupole surface modes

The anisotropy introduces a splitting between the quadrupolar modes corresponding to $(n = 0, \ell = 2, m = 1)$ and to $(n = 0, \ell = 2, m = 2)$. The density profiles and the eigenfrequencies are as follows:

$$n_1^{(0,2,1)}(\rho, z; \omega) = (1 - \rho^2 - \lambda^2 z^2)^{1/2} \rho z \exp(i\varphi),$$

$$\omega = (1 + \lambda^2)^{1/2} \omega_f, \quad (4.3)$$

and

$$n_1^{(0,2,2)}(\rho, z; \omega) = (1 - \rho^2 - \lambda^2 z^2)^{1/2} \rho^2 \exp(2i\varphi),$$

$$\omega = 2^{1/2} \omega_f. \quad (4.4)$$

In these equations φ is the angular variable in the transverse plane. The two eigenfrequencies become degenerate in the spherical limit, where $\omega = 2^{1/2} \omega_f$.

4.3 Monopole and quadrupole coupled modes

The anisotropy introduces a coupling between the modes described by the quantum numbers $(n = 1, \ell = 0, m = 0)$ and $(n = 0, \ell = 2, m = 0)$. The associated density fluctuations are

$$n_1^{(1,0,0)}(\rho, z; \omega) = (1 - \rho^2 - \lambda^2 z^2)^{1/2} (a - \rho^2 - z^2) \quad (4.5)$$

and

$$n_1^{(0,2,0)}(\rho, z; \omega) = (1 - \rho^2 - \lambda^2 z^2)^{1/2} (2z^2 - \rho^2) \quad (4.6)$$

where a is to be determined from a normalization condition. From the coupled eigenmode equations we find the eigenfrequencies as

$$\omega = 3^{-1/2} \omega_f [4\lambda^2 + 5 \pm (16\lambda^4 - 32\lambda^2 + 25)^{1/2}]^{1/2}. \quad (4.7)$$

The dispersion relation in equation (4.7) also holds for these modes in a degenerate Bose gas and in a classical gas [21].

The above examples should be sufficiently illustrative of the procedure to be followed in determining the small-amplitude oscillations in the case of axial anisotropy.

5 Collisionless regime

We return to the case of spherical confinement for the purpose of discussing the role of collisions in the dynamics of the fermion cloud. We have already remarked in Section 2 that in the homogeneous limit equation (2.14), which was derived with the help of the expression (2.11) for the kinetic stress tensor, describes first sound in a collisional regime where the local Fermi sphere executes a “breathing” oscillation. In the so-called collisionless (or zero sound) regime, on the other hand, the relaxation time of density fluctuations in the homogeneous Fermi fluid is long compared with the period of the eigenmode and the local Fermi sphere undergoes anisotropic deformations (see for instance Pines and Nozières [17]). In the limit of vanishing coupling the speed of zero sound tends to the Fermi velocity v_F (compared with the value $3^{-1/2} v_F$ for the velocity of first sound) and the deformation of the Fermi surface reduces to a small bump in the direction of the propagation vector. The collective zero-sound mode involves in this limit only a small number of quasiparticles and essentially propagates at their velocity.

For a discussion of the dynamics of a magnetically confined fermion cloud in a regime where collisions have little influence, we resort to an equation for the Wigner distribution function having the form of a Boltzmann equation with the collision integral set to zero, that is

$$\partial_t f_{\text{p}}(\mathbf{R}, t) + \frac{\mathbf{P}}{m} \cdot \nabla_{\mathbf{R}} f_{\text{p}}(\mathbf{R}, t) - \nabla_{\mathbf{R}} V_0(\mathbf{R}) \cdot \nabla_{\mathbf{p}} f_{\text{p}}(\mathbf{R}, t) = 0. \quad (5.1)$$

This equation is in fact analogous to the Landau transport equation for quasiparticles in a homogeneous Fermi liquid in the collisionless regime and is expected to be valid down to very low temperature provided that the cloud contains a large number of fermions.

In searching for solutions of equation (5.1), we focus on the shape-deformation modes which can be monitored by measuring the mean square radius of the cloud as a function of time. Within this class of modes we adopt a semiclassical variational *ansatz* as already proposed

by Bijlsma and Stoof [22] for a bosonic cloud. This amounts to choosing $f_p(\mathbf{R}, t)$ in the form of the equilibrium solution $\Phi_{\text{eq}}(\mathbf{p}, \mathbf{R})$ after rescaling both the space and the momentum variable through time-dependent factors in each geometric direction. More precisely, for $i = x, y, z$ we introduce variational parameters $\alpha_i(t)$ describing the deviation of the mean square sizes of the cloud from their equilibrium values,

$$\alpha_i(t) = [\langle R_i^2(t) \rangle / \langle R_i^2 \rangle_{\text{eq}}]^{1/2}. \quad (5.2)$$

The form of $f_p(\mathbf{R}, t)$ is then taken as

$$f_p(\mathbf{R}, t) = (c_x c_y c_z) \times \Phi_{\text{eq}}(\{c_i^{1/2} \alpha_i(p_i - m R_i \dot{\alpha}_i / \alpha_i)\}, \{c_i^{1/2} R_i / \alpha_i\}), \quad (5.3)$$

the time-independent coefficients c_i being introduced in equation (5.3) in order to ensure that the initial momentum distribution is isotropic. The equilibrium form of the Wigner distribution for the degenerate Fermi gas subject to spherical confinement can in turn be chosen as a generalized Fermi sphere,

$$\Phi_{\text{eq}}(\mathbf{p}, \mathbf{R}) = \theta \left(\frac{p^2}{2m} + \frac{1}{2} m \omega_f^2 R^2 - \mu_f \right), \quad (5.4)$$

μ_f being the chemical potential. Of course, equation (5.4) yields the same equilibrium density as that reported earlier in equation (2.13). We emphasize that in the present treatment we are assuming that, even though collisions have negligible influence on the dynamics of the cloud, they must have been active in the past in order to bring the cloud to the state of thermal equilibrium.

The equations of motion for $\alpha_i(t)$ in each spatial direction are obtained by taking the moments of equation (5.1) with respect to $R_i p_i$ [22]:

$$\ddot{\alpha}_i(t) + \omega_f^2 \alpha_i(t) = \omega_f^2 \alpha_i^{-3}(t). \quad (5.5)$$

It is worth noticing that for a non-interacting gas equation (5.5) follows directly from the dynamical scaling *ansatz* and does not require a specific assumption on the shape of the variational function.

The solution of the non-linear equation of motion (5.5), under the initial condition $\alpha_i(0) = 1$, is

$$\alpha_i(t) = (2^{1/2} \omega_f)^{-1} \times [c + (c^2 - 4\omega_f^4)^{1/2} \sin(\pm 2\omega_f t - \varphi)]^{1/2}, \quad (5.6)$$

with $\sin \varphi = [(c - 2\omega_f^2)/(c + 2\omega_f^2)]^{1/2}$ and the constant c determined by the initial velocity according to

$$\dot{\alpha}(0) = \pm (c - 2\omega_f^2)^{1/2}. \quad (5.7)$$

A more direct contact can be made with the results reported in the preceding sections for the gas in the collisional regime by linearizing equation (5.5). In such small-oscillations regime equation (5.5) becomes an equation for the linearized mean square radius $\alpha_i^{(1)}(t)$, which reads

$$\ddot{\alpha}_i^{(1)}(t) + 4\omega_f^2 \alpha_i^{(1)}(t) = 0. \quad (5.8)$$

Evidently one obtains an oscillation of the cloud at frequency $2\omega_f$, corresponding to the frequency of a single-particle excitation in the cloud.

As a final remark we point out that, while equation (5.1) satisfies the generalized Kohn theorem, a more sophisticated variational *ansatz* would be needed to extract from it the sloshing mode [22].

6 Summary and concluding remarks

In summary, we have presented an investigation of the collective excitations of a spin-polarized Fermi gas confined in an external harmonic trap. We have obtained analytic results for the small-amplitude eigenmodes in a dynamical regime corresponding to first sound in the bulk and proposed a Landau-Boltzmann approach to treat the dynamics of the confined gas in a dynamical regime corresponding to zero sound in the bulk. In spite of the assumed negligible role of interactions between the particles of the gas in contributing to a mean field sustaining a collective oscillation, we have seen that the Pauli pressure alone can determine a rather rich dynamical behaviour.

Trapped mixtures of bosonic and fermionic species are expected to become accessible to experiment in the near future. The investigation of collective excitations in such mixtures seems a natural direction of development for the present study. The density profiles of the mixtures have been determined theoretically both in the ground state at zero temperature [15] and in the equilibrium state at finite temperature [13]. From these studies one can easily envisage situations in which the dynamics of a boson-fermion mixture should be amenable to analytical approaches. Theoretical progress in this area will be reported elsewhere.

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Appendix A: Collective excitations in Fermi versus Bose-condensed clouds

In this appendix we first show how the dynamics of a dilute Bose-condensed cloud at zero temperature can be studied within the Wigner-distribution formalism that we have used for the spin-polarized Fermi gas (see also [14]). We thereby derive the analogue of the equation of motion (2.14) for density fluctuations in the Bose-condensed cloud and point out that for harmonic confinement it coincides with that obtained by Stringari [7] in a hydrodynamic approach to the strong coupling limit. We then comparatively discuss the nature of the solutions of the two equations of motion for the Fermi gas and for the Bose-condensed cloud.

A.1 Equation of motion for density fluctuations in a Bose-condensed cloud

In the case of a Bose-condensed system at zero temperature, and neglecting depletion due to the interactions, the one-body density matrix introduced in equation (2.1) reduces to the simple product of two condensate wave functions,

$$\rho(\mathbf{x}, \mathbf{x}'; t) = \Phi^*(\mathbf{x}, t)\Phi(\mathbf{x}', t). \quad (\text{A.1})$$

The generalized hydrodynamic equations (2.7, 2.8) are thereby closed and the kinetic stress tensor is explicitly given by

$$\Pi_{ij}(\mathbf{R}, t) = \frac{\hbar^2}{m} \left[\frac{\partial^2}{\partial r_i \partial r_j} \left\{ \Phi^* \left(\mathbf{R} - \frac{\mathbf{r}}{2}, t \right) \Phi \left(\mathbf{R} + \frac{\mathbf{r}}{2}, t \right) \right\} \right]_{r=0}. \quad (\text{A.2})$$

However, for a Bose condensate with repulsive interactions in the strong coupling limit the kinetic stress tensor becomes in fact negligible compared with the confinement and interaction energy terms. The linearized equation of motion for density fluctuations becomes [23, 24]

$$\begin{aligned} \partial_t^2 n_1(\mathbf{R}, t) &= \frac{1}{m} \int d\mathbf{x}' \int d\mathbf{x}'' [n_0(\mathbf{R}) \nabla_{\mathbf{R}}^2 \delta(\mathbf{R} - \mathbf{x}') \\ &+ \nabla_{\mathbf{R}} n_0(\mathbf{R}) \cdot \nabla_{\mathbf{R}} \delta(\mathbf{R} - \mathbf{x}')] \nu(\mathbf{x}', \mathbf{x}'') n_1(\mathbf{x}'', t), \end{aligned} \quad (\text{A.3})$$

where $\nu(\mathbf{x}', \mathbf{x}'')$ is the interparticle potential.

Assuming contact interactions, *i.e.* $\nu(\mathbf{x}', \mathbf{x}'') = g\delta(\mathbf{x}' - \mathbf{x}'')$ with $g = 4\pi\hbar^2 a/m$ where a is the s -wave scattering length, equation (A.3) immediately yields

$$\partial_t^2 n_1(\mathbf{R}, t) = \frac{g}{m} \nabla_{\mathbf{R}} \cdot [n_0(\mathbf{R}) \nabla_{\mathbf{R}} n_1(\mathbf{R}, t)]. \quad (\text{A.4})$$

This yields the equation of motion derived by Stringari [7] when the Thomas-Fermi solution is used for the equilibrium density profile $n_0(\mathbf{R})$.

Before proceeding we wish to emphasize that the nature of the Thomas-Fermi approximation is very different in the two cases that we are considering. In the spin-polarized Fermi gas the interactions are negligible, so that the kinetic term provides the only energy scale. On the other hand, in a mesoscopic Bose condensate the interactions are dominant even at zero temperature. Nevertheless, the form of equation (A.4) is essentially similar to that of equation (2.14).

A.2 Comparative discussion of the solutions of equations (A.4, 3.1)

The solutions of equation (A.4) for the Bose-condensed cloud vanish outside the cloud radius R_b and present a discontinuity at R_b [7]. Indeed, by the Fuchs method used in Section 3 one may prove that it is not possible to impose

continuity at R_b . The discontinuity is physically acceptable in view of the fact that the kinetic energy term has been set as negligible in taking the strong-coupling limit.

On the other hand, equation (3.1) for the Fermion cloud admits, in addition to the solution that we have presented in the main text, a non-vanishing solution outside the cloud radius R_f . This solution is

$$n_1(\mathbf{R}, \omega) = C\vartheta(y-1)y^{-[3+(\omega/\omega_f)^2]}. \quad (\text{A.5})$$

Evidently, such a finite external solution going to zero only for $y \rightarrow \infty$ as in equation (A.5) is inconsistent with the assumption of small-amplitude oscillations of the cloud. Nevertheless, if for a moment one admits it and asks that the constant C be finite and determined by imposing continuity at the boundary with the internal solution (as necessary in a system where the kinetic energy is dominant), one finds a dispersion relation which is different from that given in equation (3.6), that is

$$\omega = \omega_f \left\{ \frac{1}{3} [\ell + 4n(n + \ell + 1)] - 1 \right\}^{1/2}. \quad (\text{A.6})$$

It is evident that the modes with $n = 0$ and $\ell \leq 3$ are suppressed, and in particular the sloshing mode which is expected from the generalized Kohn theorem is absent. We have for this reason discarded this second solution and, by imposing that the internal solution vanishes continuously at the boundary (*i.e.* $C = 0$), found the dispersion relation reported in equation (3.6) in the main text. We also remark that each physically acceptable eigenfrequency of the cloud, with increasing n at fixed ℓ in equation (3.6), lies between two of the unphysical frequencies given by equation (A.6).

In conclusion, the different roles played by the kinetic energy in the Bose condensate and in the Fermi gas lead to very different forms of the density fluctuations in these two systems. In the former system the amplitude of the fluctuations vanishes discontinuously at the boundary of the cloud in the strong-coupling limit, whereas in the latter it vanishes continuously at the boundary.

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